ON THE NONSYMMETRIC HYPERSONIC FLOW AROUND A CIRCULAR CONE

(O NESIMMETRICHNOM GIPERZVUKOVOM OBTEKANII KRUGOVOGO KONUSA)

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B.M. BULAKH
(Saratov)

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Gonor [1] has considered a nonviscous hypersonic flow around a circular cone with an angle of attack α. He sought a solution in the form of a power series in $\varepsilon = (\gamma - 1)/(\gamma + 1)$ (where γ is the adiabatic index), and obtained the first term of the expansion in closed parametric form. In the present work it is shown that near the cone surface there exists a "vortex layer" in which the solution can not be approximated by means of a partial sum of the power series in ε. Because of this, the theory of Gonor does not offer the possibility of determining the velocity components on the surface of the cone. Here it is shown how the velocity components can be found on the cone surface with a precision of $O(\varepsilon)$ by starting out with Gonor's solution.

1. Let us consider a stationary flow around a circular cone with semi-vertex angle θ_k. The flow is that of a nonviscous homogeneous gas. It has a hypersonic velocity with an angle of attack α in a spherical system of coordinates r, θ, φ whose axis coincides with the cone axis (see figure).

Let us denote by u, v and w the components of the velocity vector of the gas particles in the directions of increasing r, θ and φ; p and ρ will denote pressure and density, respectively. Quantities which are related to the undisturbed flow we shall mark with a right superscript o (V° is the velocity, M° is the Mach number of the undisturbed flow).

We shall treat this problem in the frame of the theory of conical flow, when u, v, w, p and ρ do not depend on r. The equations of continuity, momentum, and energy can be expressed in this case in the following form:

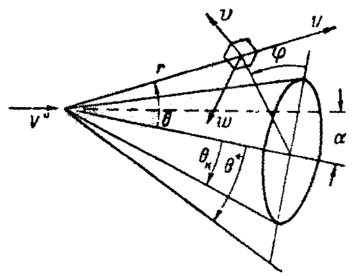
$$2\rho u \sin\theta + (\rho v \sin\theta)_\theta + (\rho w)_\varphi = 0 \qquad (1.1)$$

$$v u_\theta + w \csc\theta u_\varphi - v^2 - w^2 = 0 \qquad (1.2)$$

$$v v_\theta + w \csc\theta v_\varphi + \frac{1}{\rho} p_\theta + uv - \frac{w^2}{\tan\theta} = 0 \qquad (1.3)$$

$$\frac{u^2 + v^2 + w^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{V^{\circ 2}}{2} + \frac{\gamma}{\gamma - 1}\frac{p^\circ}{\rho^\circ} \qquad (1.4)$$

$$v\left(\frac{p}{\rho^\gamma}\right)_\theta + w \csc\theta \left(\frac{p}{\rho^\gamma}\right)_\varphi = 0 \qquad (1.5)$$



The subscripts θ and φ indicate differentiation, γ is the adiabatic index; the gas is assumed to be an ideal gas. After the elimination of ρ, Equation (1.1) can be represented in the form (a is the velocity of sound):

$$(a^2 - v^2)\sin\theta v_\theta = (w^2 - a^2) w_\varphi + vw(\sin\theta w_\theta + v_\varphi) + u\,(v^2 + w^2 - 2a^2)\sin\theta - a^2 v \sin\theta$$

$$\left(a^2 = a^{\circ 2} - \frac{\gamma - 1}{2}(u^2 + v^2 + w^2 - V^{\circ 2})\right) \qquad (1.6)$$

2. In solving this problem, Gonor [1] used G.G. Chernyi's method of expanding the solution in powers of $\varepsilon = (\gamma - 1)/(\gamma + 1)$ (under the condition that $\beta = 1/M^{\circ 2}(\gamma - 1)$ be of the order of 1 or lower when $\varepsilon \to 0$).

In the region between the cone surface and the shock wave, the solution was sought in the form

$$u = u_0 + \varepsilon u_1 + \ldots,\quad v = \varepsilon v_0 + \varepsilon^2 v_1 + \ldots,\quad w = w_0 + \varepsilon w_1 + \ldots \qquad (2.1)$$

$$p = p_0 + \varepsilon p_1 + \ldots,\quad \rho = \frac{1}{\varepsilon}\rho_0 + \rho_1 + \ldots$$

The chosen independent variables were φ, ψ, where ψ = const on the curves of constant entropy (θ was expressed also in the form $\theta = \theta_k + \varepsilon\theta_0 + \varepsilon^2\theta_1 + \ldots$). It is expedient, for what follows, to introduce variables of the "boundary layer" directly into the physical space:

$$\vartheta = (\theta - \theta_k)\varepsilon^{-1},\quad \varphi = \varphi \qquad (2.2)$$

3. We shall show that on the cone surface the velocity components u and w, determined by (2.1), do not satisfy Equation (1.2), which for $\theta = \theta_k$ can be written in the form

$$w = \frac{1}{\sin\theta_k} u_\varphi$$

Substituting into this the values given in (2.1), we obtain

$$w_0 = \frac{1}{\sin\theta_k} u_{0\varphi}, \qquad w_1 = \frac{1}{\sin\theta_k} u_{1\varphi}, \ldots \tag{3.1}$$

In Gonor's theory [1], $w_0 = 0$, when $\theta = \theta_k$, and $u_{0\varphi} \neq 0$, and, hence, (3.1) is not satisfied. The inapplicability of the expansion (2.1) in the neighborhood of the cone is also revealed by the fact that in Gonor's solution the surface of the cone is not a surface of constant entropy.

4. In order to find u and w when $\theta = \theta_k$, we use the method by which Willett [2] determined u and w on the cone in a supersonic flow. The surface of the cone is a stream surface which begins on the shock wave when $\varphi = \pi$. In view of this, the entropy on the cone surface must have the same value as it has on the shock wave when $\varphi = \pi$ (see, for example, [3]).

If one sets θ (for the shock wave with $\varphi = \pi$) equal to $\theta^* = \theta_k + \varepsilon\vartheta^* + \ldots$, considers the conditions at the discontinuity when $\varphi = \pi$, takes account of the fact that p/ρ^γ = const on the considered flow surface, and eliminates ρ from Bernoulli's equation (1.4), then one obtains, with $\theta = \theta_k$, the equation

$$u^2 + w^2 + \frac{2\gamma}{\gamma+1}\left(\frac{p}{\rho^\circ}\right)^{\frac{\gamma-1}{\gamma}}\left(\frac{V^{\circ 2}}{\gamma}\right)^{\frac{1}{\gamma}}\left[\frac{2\gamma}{\gamma+1}\sin^2(\alpha+\theta^*) - \varepsilon^2\beta(\gamma+1)\right]^{\frac{1}{\gamma}} \times$$

$$\times\left[1 + \frac{2\beta}{\sin^2(\alpha+\theta^*)}\right] = (1+2\beta)V^{\circ 2} \qquad \left(\beta = \frac{1}{M^{\circ 2}(\gamma-1)}, \ \varepsilon = \frac{\gamma-1}{\gamma+1}\right) \qquad (v=0) \tag{4.1}$$

Let us assume that on the cone surface, u, w and p can be expressed in the form

$$u = u_0^\times + \varepsilon u_1^\times + \ldots, \quad w = w_0^\times + \varepsilon w_1^\times + \ldots, \quad p = p_0^\times + \varepsilon p_1^\times + \ldots \tag{4.2}$$

Substituting (4.2) and $\theta^* = \theta_k + \varepsilon\vartheta^* + \ldots$ into (4.1), expanding the result in powers of ε, and equating coefficients of like powers of ε, we obtain

$$u_0^{\times 2} + w_0^{\times 2} = V^{\circ 2}\cos^2(\alpha+\theta_k) \tag{4.3}$$

$$w_0^\times w_1^\times + u_0^\times u_1^\times + V^{\circ 2}\Big\{\vartheta^*\cos(\alpha+\theta_k)\sin(\alpha+\theta_k) + \tag{4.4}$$

$$+ [\sin^2(\alpha+\theta_k) + 2\beta]\ln\frac{p_0^\times}{\rho^\circ V^{\circ 2}\sin^2(\alpha+\theta_k)}\Big\}$$

Substituting (4.2) into (1.2) with $\theta = \theta_k$, we obtain in an analogous manner

$$w_0^{\times} = \frac{1}{\sin\theta_k}\, u_{0\varphi}^{\times}, \qquad w_1^{\times} = \frac{1}{\sin\theta_k}\, u_1^{\times}\varphi \tag{4.5}$$

Substitution of (4.5) into (4.3) leads to the differential equation (with $\theta = \theta_k$)

$$u_0^{\times 2} + \frac{1}{\sin^2\theta_k} u_{0\varphi}^{\times 2} = V^{\circ 2}\cos^2(\alpha + \theta_k) \tag{4.6}$$

The unique solution of (4.6) satisfying the condition

$$w_0^{\times} = \frac{1}{\sin\theta_k}\, u_{0\varphi}^{\times} = 0 \quad \text{for } \varphi = 0, \pi$$

will be

$$u_0^{\times} = V^\circ \cos(\alpha + \theta_k), \qquad w_0^{\times} \equiv 0 \tag{4.7}$$

From (4.4), (4.5) and (4.7) we now obtain

$$u_1^{\times} = V^\circ \left[-\vartheta^* \sin(\alpha + \theta_k) + \frac{\sin^2(\alpha + \theta_k) + 2\beta}{\cos(\alpha + \theta_k)} \ln \frac{\rho^\circ V^{\circ 2} \sin^2(\alpha + \theta_k)}{p_0^{\times}} \right] \tag{4.8}$$

$$w_1^{\times} = -V^\circ \frac{\sin^2(\alpha + \theta_k) + 2\beta}{\sin\theta_k \cos(\alpha + \theta_k)} \frac{p_{0\varphi}^{\times}}{p_0^{\times}}$$

From (4.7) it follows that the quantity w_0, with $\theta = \theta_k$, is given correctly by Gonor's solution, but this is not the case for u_0.

5. Let us investigate the behavior of the solution near the cone surface. From (4.7), (4.8) and (1.6) it follows that when $\theta = \theta_k$

$$v_\theta = -2u_0^{\times} + O(\varepsilon)$$

Therefore, in the neighborhood of $\theta = \theta_k$

$$v = -2u_0^{\times}(\theta - \theta_k) + O[\varepsilon(\theta - \theta_k)] + o\,(\theta - \theta_k) = -2V^\circ \cos(\alpha + \theta_k)\,\varepsilon\vartheta + o\,(\varepsilon\vartheta) \tag{5.1}$$

while w can be expressed in the form

$$w = w_0'(\vartheta, \varphi) + \varepsilon w_1^{\times}(\varphi) + o(\varepsilon) \tag{5.2}$$

where $w_0'(0, \varphi) \equiv 0$. (If Gonor's theory determines w_0 correctly not only when $\theta = \theta_k$ but also in some neighborhood of $\theta = \theta_k$, then $w_0' = w_0$.)

We now introduce into our discussion the quantity

$$S = \frac{p}{(\varepsilon\rho)^\gamma}$$

Substituting (5.1) and (5.2) into (1.5), dropping the small terms, and passing from θ to ϑ, we obtain an equation for S

$$\vartheta S_{\vartheta} + [\varepsilon g(\varphi) + h(\vartheta, \varphi)] S_{\varphi} = 0 \tag{5.3}$$

$$\left(g(\varphi) = - \frac{w_1^{\times}(\varphi)}{2V \cos(\alpha + \theta_k)}, \quad h(0, \varphi) \equiv 0 \right)$$

The general solution (5.3) has the structure

$$S = f\left[\vartheta^{\varepsilon} \exp\left(- \int \frac{d\varphi}{g} \right) Q \right] \tag{5.4}$$

Here f is an arbitrary function while the function Q satisfies the equation

$$\vartheta Q_{\vartheta} + (\varepsilon g + h) Q_{\varphi} - \frac{h}{g} Q = 0 \tag{5.5}$$

with the boundary condition $\vartheta = 0$, $Q = 1$.

From (5.4) and (5.5) it is clear that for small values of ϑ, the function S can not be approximated by a finite partial sum of a power series in ε, because such an approximation would not be valid for ϑ^{ε}, or for Q.

For example, every partial sum of the series

$$\vartheta^{\varepsilon} = e^{\varepsilon \ln \vartheta} = 1 + \varepsilon \ln \vartheta + \frac{1}{2} (\varepsilon \ln \vartheta)^2 + \ldots$$

becomes infinite when $\vartheta = 0$, while $\vartheta^{\varepsilon} = 0$ when $\vartheta = 0$.

The possibility of expanding S as a convergent series in ε is related to the possibility of expanding ϑ^{ε} and Q in such series. For ϑ^{ε} this is possible if $\vartheta = O(\varepsilon)$, because $\varepsilon \ln \varepsilon$ is small for small ε; the behavior of Q is determined by the function h.

If one assumes that $w_0' = w_0$, then $h \sim \sqrt{\vartheta}$, and with $\vartheta = O(\varepsilon)$, the function Q can be expressed as the series

$$Q = Q_0 + \varepsilon Q_1 + \varepsilon^2 Q_2 + \ldots$$

For S we have the following expansion in powers of ε:

$$S = S_0 + \varepsilon S_1 + \ldots = f\left[\exp\left(- \int \frac{d\varphi}{g} \right) Q_0 \right] + \varepsilon O(\ln \vartheta) + \ldots \tag{5.6}$$

A study of the equations of the system (1.1) to (1.5) reveals that there are no obstacles to the expansion of the other solution quantities in power series of ε when $\vartheta = O(\varepsilon)$. One may, therefore, expect the expansion (2.1) to represent the solution outside the vortex layer of thickness $\vartheta = O(\varepsilon)$, and one may assume that, when $\vartheta = O(\varepsilon)$, S can be expressed in the form

$$S = \frac{p}{(\varepsilon\rho)^{\gamma}} = \frac{p_0}{\rho_0} + \varepsilon\left(\frac{p_1}{\rho_0} - \frac{p_0\rho_1}{\rho_0^2} - \frac{p_0}{\rho_0} 2\ln\rho_0\right) + \ldots = S_0 + \varepsilon S_1 + \ldots \quad (5.7)$$

From (5.6) and (5.7) we obtain

$$f\left[\exp\left(-\int\frac{d\varphi}{g}\right)Q_0\right] = \frac{p_0}{\rho_0} \text{ when } \vartheta = 0, \qquad \rho_1 \sim \ln\vartheta \text{ when } \vartheta \to 0$$

One finds also that

$$u_1 \sim \ln\vartheta \text{ when } \vartheta \to 0$$

If one passes from θ to ϑ in the equations (1.1) to (1.5), substitutes in them the expansion (2.1), and obtains the equations for u_1, v_1, w_1, p_1 and ρ_1, then one can establish that if $w_1 \neq 0$ when $\vartheta = 0$, $\rho_1 \sim \log\vartheta$, and $u_1 \sim \log\vartheta$ when $\vartheta \to 0$. This confirms the above assertion.

From Equation (1.3) one can see that $p_\theta = O(\varepsilon)$ in the vortex layer. Therefore, the pressure change through the vortex layer of thickness $\theta - \theta_k = \varepsilon\ \vartheta = O(\varepsilon^2)$ is of the order $O(\varepsilon^3)$.

6. From the qualitative analysis of Section 5, and from the similarity between the structures of the vortex layers in supersonic and hypersonic flows (see, for example, [4]) one can expect that the expansion (2.1) gives the correct values of p and w to within an error of order $O(\varepsilon)$, and that this expansion represents the solution outside a vortex layer of thickness $\theta - \theta_k = O(\varepsilon^2)$.

If one accepts what has been said above, then $p_0{}^{x} = p_0$ when $\theta = \theta_k$, and Formulas (4.2), (4.7) and (4.8) determine u and w on the surface of the cone with a precision of $O(\varepsilon)$, whereby ϑ^* is given by the formula

$$\vartheta^* = \frac{\sin^2\theta_k\cos(\alpha+\theta_k)\left[\sin^2(\alpha+\theta_k)+2\beta\right]}{\sin^2\alpha\sin(\alpha+\theta_k)}\times \quad (6.1)$$
$$\times\left\{\left[1+\frac{\sin\alpha}{\sin\theta_k\cos(\alpha+\theta_k)}\right]\ln\left[1+\frac{\sin\alpha}{\sin\theta_k\cos(\alpha+\theta_k)}\right]-\frac{\sin\alpha}{\sin\theta_k\cos(\alpha+\theta_k)}\right\}$$

This formula can be obtained from Gonor's [1] results by elementary means but cumbersome computations. When $\alpha \to 0$, the Formulas (4.7), (4.8) and (6.1) go over into well-known formulas for a cone with $\alpha = 0$ (see, for example, [5]). We note that the vortex layer exists also near the surface of any conical body. This is related to the fact that w_0 becomes zero at the surface of a body [6]. The velocity components on the surface of the body can be computed in an analogous way.

Not until after this paper had been written did the author become acquainted with a recent article of Cheng [7] where analogous results are given, but only for the cases of small and intermediate angles of attack (Cheng expanded the solution in power series of ε and $\sigma = \sin \alpha / \sin \theta_k$ and restricted himself to terms of order $O(\sigma^2)$.

BIBLIOGRAPHY

1. Gonor, A.L., Obtekanie konusa pod uglom ataki s bol'shoi sverkhzvukovoi skorst'iu (Supersonic flow around a cone with an angle of attack). *Izv. Akad. Nauk SSSR, OTN* No. 7, 1958.

2. Willett, J.E., Supersonic Flow at the Surface of a Circular Cone at Angle of Attack. *I.A.S.* Vol. 27, No. 12, 1960.

3. Ferri, A., Supersonic Flow Around Circular Cones at Angle of Attack. *NACA, Report* No. 1045, 1951.

4. Bulakh, B.M., Sverkhzvukovoi potok okolo naklonennogo krugovogo konusa (Supersonic flow around an inclined circular cone). *PMM* Vol. 26, No. 2, 1962.

5. Chernyi, G.G., Obtekanie tel ideal'nym gazom pri bol'shoi sverkhzvukovoi skorosti (Supersonic flow of an ideal gas around conical bodies). *Izv. Akad. Nauk SSSR, OTN* No. 6, 1957.

6. Gonor, A.L., Obtekanie konicheskikh tel pri dvizhenii gaza s bol'shoi sverkhzvukovoi skorosti (Supersonic gas flow around conical bodies). *Izv. Akad. Nauk SSSR OTN* No. 1, 1959.

7. Cheng, H.K., Hypersonic flows past a yawed circular cone and other pointed bodies. *J. Fluid Mech.* Vol. 12, p. 2. February, 1962.

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